On the Sobolev quotient in CR geometry

Joint work with J.H.Cheng and P.Yang

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Taipei, Jan. 20, 2018

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If R_g is the scalar curvature, setting $\tilde{g}(x) = \lambda(x)g(x) = u(x)^{\frac{4}{n-2}} g(x)$, $u(x)$ one has to find on M a positive solution of

$$(Y) \qquad -c_n \Delta u + R_g \, u = \overline{R} \, u^{\frac{n+2}{n-2}}; \qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

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Considering $\overline{R}$ as a Lagrange multiplier, one can try to find solutions by minimizing the *Sobolev-Yamabe quotient*

$$Q_{SY}(u) = \frac{\int_M \left(c_n |\nabla u|^2 + R_g u^2\right) dV}{\left(\int_M |u|^{2^*} dV\right)^{\frac{2}{2^*}}}; \qquad 2^* = \frac{2n}{n-2}.$$

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The *Sobolev-Yamabe constant* is defined as

$$Y(M,[g]) = \inf_{u \not\equiv 0} Q_{SY}(u).$$

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• Since S^n is conformal to $\mathbb{R}^n$, one has that $Y(S^n,[g_{S^n}]) = S_n$.

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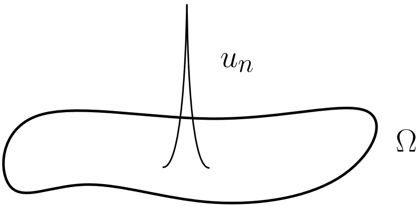
Minimizing sequences u_n tend to concentrate indefinitely inside Ω.



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- In 1984 Schoen proved that $Y(M,[g]) < S_n$ in all other cases, i.e. $n \leq 5$ or (M,g) locally conformally flat, unless $(M,g) \simeq (S^n, g_{S^n})$.

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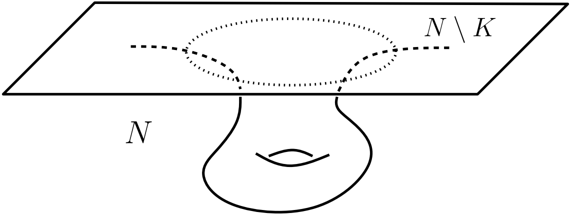

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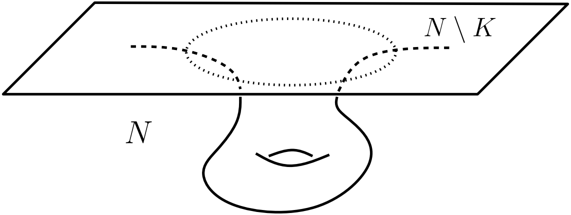

In general relativity these manifolds describe static gravitational systems.

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Then, in normal coordinates x at p, setting $y=\frac{x}{|x|^2}$ (Kelvin inversion) one has an asymptotically flat manifold in y-coordinates

$$\tilde{g}(x) \simeq \frac{dx^2}{|x|^4} \simeq dy^2, \qquad (y \text{ large}).$$

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In fact, one has

$$\frac{d}{dg} \left(R_g \, dV_g \right) [h] = - \left(h^{ij} E_{ij} + \text{div } X \right) dV_g,$$

where X is some vector field.

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If $f(x) = G_p^4 \simeq d(x,p)^{-4}$, then

$$m_{ADM} = \lim_{x \to p} \left(G_p(x) - \frac{1}{d(x,p)} \right) = A.$$

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If $R_g \geq 0$ then $m(g) \geq 0$. In case $m(g) = 0$, then (M, g) is <u>isometric</u> to the flat Euclidean space $(\mathbb{R}^3, dx^2)$.

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In 1981 Witten ('81) used Dirac's equation in a different proof, obtaining an integral formula for the mass via the Bochner-Lichnerowitz identity. Both approaches are fundamental to study manifolds with positive scalar curvature ([Gromov-Lawson, '80], [Stolz, '92]).

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- Relation to stability properties of minimal surfaces ([Carlotto, '14], [Carlotto-Chodosh-Eichmair, '15]).

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This condition is quite important for the study of biholomorphic mappings and the $\overline{\partial}$-Neumann problem ([Beals-Fefferman-Grossman, '83]).

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As before, we can define a *Sobolev-Webster quotient* and try to uniformize W as we did for the scalar curvature. For $n \geq 5$ Jerison and Lee (1989) proved the counterparts of Trudinger and Aubin's results.

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However, things start to get different in the CR case. One crucial difference between dimension three and higher is the *embeddability* of abstract CR manifolds (reference book: [Chen-Shaw, '01]).

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The only obstruction to the positivity is the first term: however by a recent theorem in [Hsiao-Yung, '15] one can kill $Z_1 Z_{\overline{1}} \beta$ starting from an approximate solution decaying sufficiently fast at ∞.

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• $m > 0$ implies that the Sobolev quotient of (M, J) is lower than that of S^3, so minimizers exist.

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A positive mass theorem

Theorem ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the CR Paneitz operator is non-negative. Let $p \in M$ and let $\tilde{\theta}$ be a blown-up of contact form at p. Then

(a) the CR mass of $(M, J, \tilde{\theta})$ is non negative;

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• Positivity is proved in higher dimensions in [Cheng-Chiu-Yang, '14] for *locally spherical manifolds*, and in [Cheng-Chiu, w.i.p.] for $n = 5$.

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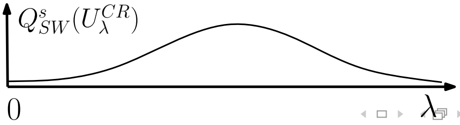
- For $|s| \neq 0$ small, the Webster quotient of the functions U^{CR}_λ has a profile of this kind, for λ in a fixed compact set of $(0, \infty)$



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Remark. The CR Sobolev quotient of S_s^3, a <u>closed</u> manifold, behaves like that of a domain in $\mathbb{R}^n$!

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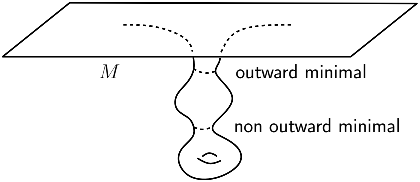
Assuming finite volume, it is done in [Jerison-Lee, '88]. However blow-up profiles may not satisfy this assumption, and Moving planes do not work. Related Liouville thms. in [Birindelli-Capuzzo Dolcetta-Cutrì, 97].

Penrose's inequality gives quantitative lower bounds on the ADM mass in terms of *outward minimal surfaces*



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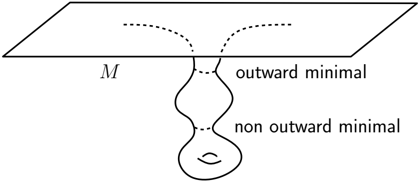

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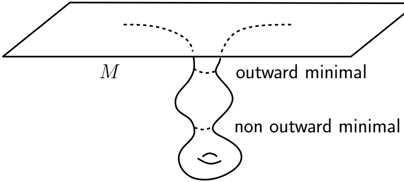

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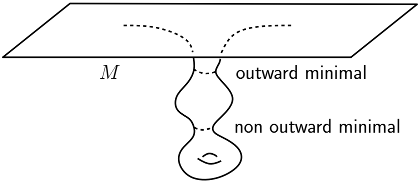

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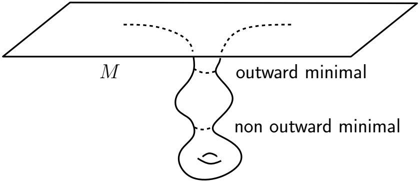

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In the CR case, is it possible always true that negative mass implies that the Sobolev quotient is not attained?

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Happy Birthday Alice and Paul!